Effectiveness and Multivalued Logics
(An extended abstract from a paper published on Journal of Symbolic logic)

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Keywords: Multi-valued logic, fuzzy logic, computability, domain theory.

1. Introduction

The recent researches in multi-valued logics usually consider the truth values inside the language by assuming, for example, that (a dense subset of) the truth values are represented by corresponding propositional constants. So, formulas as $0.4\rightarrow(\alpha\rightarrow 0.3)$ or $0.1\rightarrow(0.2\rightarrow 0.3)$ are admitted. This means that, as in classical logic, if F indicate the set of formulas in such a language, then both the available information and the derived information are represented by classical subsets of F. This entails an approach whose paradigm coincides with the one of classical logic. In particular, the notion of effectiveness is inherited from recursion theory in N in spite of fact that the main examples of multi-valued logics involve the real numbers interval [0,1]. The researches along such a line are in a very advanced state and very deep results were discovered (see, for example, Hájek 1998, Gottwald 2000, Montagna 2001). Nevertheless, in accordance with Goguen 1968/69 and Pavelka 1979, I am interested in a different way to go on in which both the hypotheses (the available information) and the related consequences (the derived information) are represented by fuzzy sets of formulas. Indeed, in my opinion, this choice gives us major chances to catch the particular nature of a reasoning from vague hypotheses as Goguen's analysis of sorites paradox shows. Moreover, it gives a more appropriate way to represent the effectiveness of the inferential processes in multi-valued logic for which the notion of endless approximation process have to play a basic role.

To introduce the basic notions of fuzzy logic, we can start from the idea that the deduction apparatus in a logic is a tool for an effective management of the available information. Then, to define a logic we have to specify:

1. a set Inf to represent the pieces of information we have to manage and an order $\leq$ in Inf to express the completeness of the information
2. a "deduction operator", $D : Inf \rightarrow Inf$ to improve the available information
3. suitable properties of *effectiveness* for the whole system.

In the case of classical logic the information is completely based on the language. Indeed, an elementary piece of information is a formula in F and a piece of information is a subset T of F. The information content of T is that *at least* all the claims in T are truth while no information we have about the remaining formulas. Then, Inf is the class $P(F)$ of all the subsets of F and the order is the inclusion relation. Once the deduction operator $D : P(F)\rightarrow P(F)$ is defined, the effectiveness of the inferential process is represented by the fact that D is an enumeration operator. So, from a decidable set of hypotheses T we derive a recursively enumerable set $D(T)$ of consequences.

Passing to fuzzy logic, we will assume that an *elementary piece* of information is a sentence α in a language together with an information (i.e. a constraint) λ on the possible truth value of α. In other words an elementary piece of information is a *signed formula* (α,λ) and a *piece of information* is a set $T \subseteq F\times L$ of signed formulas. We emphasize that λ is not the truth value of α but our (incomplete, in general) information about the truth value of α. We assume that in the class L of admissible constraints there is an order $\leq$ in such a way that $\lambda_1\leq\lambda_2$ means that the constraint λ_2 is stronger than the constraint λ_1 or, equivalently, that the information λ_1 is contained in the information λ_2. Also, we assume that L is a complete lattice with respect to $\leq$. Such an hypothesis is necessary to *"fuse"* different pieces of information on the same formula α. This means that if $X\subseteq L$ is the available set of constraints on the actual truth value of α, then $\lambda = Sup(X)$ is an unique constraint "equivalent" with X. If we admit this, then we are able to represent a piece of information in a more workable way. Indeed, we can associate any piece of information T with the functional piece $v : F \rightarrow L$ defined by setting $v(\alpha) = Sup\{\lambda \in L : (\alpha,\lambda)\in T\}$. In accordance, we will assume that Inf coincides with the lattice L^F of the L-subsets of F. This gives also a natural order among pieces of information. Also, since this order is with respect to the completeness of the information, we accept the hypothesis of monotony, and, in accordance, that the deduction operator D is a closure operator in L^F.

It remains to consider in some way the effectiveness of the inferential process in fuzzy logic and the aim of this paper is to face this question. Indeed, we will give a notion of effectiveness (and therefore of continuity) for the deduction operator and we will extend the notions of recursive enumerability and decidability to the L-subsets. This extend the previous researches (see Gerla 1987, Biacino and Gerla 1989) but in this paper we refer to the theory of effective domains as defined, for example in Smyth 1977.

2. Based continuous lattices

In this paper L denotes always a complete lattice with minimum 0 and maximum 1. A subset X of L is *upward directed*, in brief *directed*, provided that for any x, $y\in X$ there is $z\in X$ such that $x\leq z$ and $y\leq z$. A family $(x_i)_{i\in I}$ is *directed* provided that $\{x_i\in L : i\in I\}$ is directed.

Definition 2.1. Let x, $y\in L$, then we say that x *is way below* y and we write $x\ll y$ provided that, for every nonempty directed subset A of L

$$y \leq SupA \;\Rightarrow\; \text{there is } a\in A \text{ such that } x \leq a.$$

In the following we list the main properties of such a relation.

Proposition 2.2. For any x and y in L,

i) $x\ll y \Rightarrow x\leq y$

ii) $x\ll y,\ x'\leq x \;\Rightarrow x'\ll y$

iii) $x\ll y,\ y'\geq y \Rightarrow x\ll y'$

iv) $x\ll z,\ z\ll y \;\Rightarrow x\ll y$ (transitivity)

v) $0\ll y$

vi) $x\ll z,\ y\ll z \;\Rightarrow\; x\vee y\ll z$

vii) $x\ll z_1,\ y\ll z_2 \;\Leftrightarrow\; x\vee y\ll z_1\vee z_2$.

Observe that $\ll$ is different from $\leq$, in general.

Definition 2.3. A *based continuous lattice*, in brief a *based lattice*, is a structure $(L,\leq,B)$ where L is a complete lattice and B, the *basis*, is a subset of L containing 0, closed with respect to $\vee$ and $\wedge$ and such that,

$$x = Sup(\{b\in B : b\ll x\}). \tag{2.1}$$

As a consequence of the closure of B with respect to $\vee$ and by Proposition 2.2, any set $\{b\in B : b\ll x\}$ is directed. Observe that usually a basis is defined as a subset B of L such that for any $x\in L$ the set $\{b\in B : b\ll x\}$ is directed and (2.1) is satisfied. Our definition is substantially equivalent. In fact, if B is a subset of L satisfying (2.1), then by adding 0 to B and by closing B with respect to $\wedge$ and $\vee$, we obtain a new subset which is a basis in our sense.

Theorem 2.4. (Interpolation theorem). Let $(L,\leq B)$ a based lattice and assume that $x\ll y$. Then there is $b\in B$ such that $x\ll b\ll y$. As a consequence, for any directed family $(x_i)_{i\in I}$,

$$y\leq Sup_{i\in I}x_i \Rightarrow \text{there is } x_i \text{ such that } x\ll x_i.$$

In the case of chains, the structure of based lattices is very simple.

Proposition 2.5. Let L be a finite chain, then $(L,\leq,B)$ is a based lattice if and only if $B = L$. In such a case

$$x\ll y \;\Leftrightarrow\; x\leq y.$$

Let L be complete chain and B a dense subset of L. Then $(L,\leq,B)$ is a based lattice such that

$$x \ll y \Leftrightarrow \text{either } x = 0 \text{ or } x<y.$$

In particular, let U be the complete lattice defined by the real numbers interval [0,1] and denote by $U_Q = U\cap Q$ the set of rational numbers in U. Then $(U,\leq,U_Q)$ is a based lattice.

3. Effective lattices, semi-decidable elements

The notion of effective lattice enables us to give an analogous of the notion of recursively enumerable subset.

Definition 3.1. An *effective continuous lattice* (*in brief an effective lattice*) is a based lattice $(L,\leq,B)$ such that there is an enumeration $(b_n)_{n\in N}$ of B in such a way that
- the set $\{(n,m)\in N^2 : b_n \ll b_m\}$ is recursively enumerable
- there are two recursive maps $join : N\times N\to N$ and $meet : N\times N\to N$ such that

$$b_n\vee b_m = b_{join(n,m)} \ , \ b_n\wedge b_m = b_{meet(n,m)}.$$

In brief, an effective lattice is a based lattice such that $\ll$ is recursively enumerable in B and $\vee$, $\wedge$ are computable in B. In any effective lattice the relation $b_n\leq b_m$ is decidable. In fact

$$b_n\leq b_m \Leftrightarrow b_n\wedge b_m = b_n \Leftrightarrow b_{join(n,m)} = b_n \Leftrightarrow join(n,m) = n.$$

Proposition 3.2. Any finite chain L is an effective lattice with respect to the basis L. The interval U is an effective lattice with respect to the basis U_Q.

Now we are able to give the first main definition in the theory of effective domains.

Definition 3.3. We say that an element x in an effective lattice $(L, \leq, B)$ is *semi-decidable* if the cut $\{n\in N : b_n\ll x\}$ is recursively enumerable.

In particular, any $b\in B$ is semi-decidable. Since by the interpolation theorem $\{n\in N : b_n\ll 1\} = \{n\in N :$ there is m such that $b_n\ll b_m\}$, 1 is semi-decidable, too. If L is finite lattices, all the elements are semi-decidable. If $L = U$, $x\in U$ is semi-decidable if and only if $\{r\in U_Q : r<x\}$ is recursively enumerable.

Proposition 3.4. Let $(L,\leq,B)$ be an effective lattice, then the following are equivalent:

i) x is semi-decidable

ii) a recursive function $f : N \to N$ exists such that $(b_{f(n)})_{n\in N}$ is a $\ll$-chain and

$$x = Sup_{n\in N} b_{f(n)}. \tag{3.1}$$

iii) a recursive function $f : N \to N$ exists such that $(b_{f(n)})_{n\in N}$ is a chain and satisfies (3.1).

iv) a recursive function $f:N\to N$ exists such that $(b_{f(n)})_{n\in N}$ is directed and satisfies (3.1).

v) a recursive function $f : N \to N$ exists such that $(b_{f(n)})_{n\in N}$ satisfies (3.1).

Proposition 3.5. There is an effective coding $w_1, w_2\ldots$ for the class $Sem(L)$ of all the semi-decidable elements of an effective lattice. Also $Sem(L)$ is a lattice and two recursive maps exist $h : N\times N\to N$ and $k : N\times N\to N$ such that

$$w_{h(n,m)} = w_n\vee w_m \quad \text{and} \quad w_{k(n,m)} = w_n\wedge w_m.$$

Proof. Given $i \in N$, let φ_i be the partial recursive function with index i. Also, denote by $Pr_1: N \to N$ and $Pr_2:N\to N$ two computable functions such that $(Pr_1,Pr_2):N\to N\times N$ is one-one. Then we define $\varphi(i,n)$ by setting
- $\varphi(i,n) = 0$ if φ_i is not convergent in fewer that $Pr_1(n)$ steps given the input $Pr_2(n)$
- $\varphi(i,n) = \varphi_i(Pr_2(n))$ otherwise.

The function φ is total recursive and by the *s-m-n*-theorem there is a recursive function h such that $\varphi(i,n) = \varphi_{h(i)}(n)$. Moreover $range(\varphi_{h(i)}) = range(\varphi_i)\cup\{0\}$. We denote by w_i the semi-decidable element such that $w_i = Sup_{n\in N} b_{\varphi_{h(i)}(n)}$. If x is any semi-decidable element and φ_i a total recursive function such that $x = Sup_{n\in N} b_{\varphi_i(n)}$, then $x = w_i$.

The second basic definition in effective domain theory is the one of computable function. Recall that, given two complete lattices L and L', a function $f : L \to L'$ is *continuous* provided that

$$f(SupX) = Supf(X)$$

for any directed class X of elements of L.

Definition 3.6. Let $(L,\leq,B)$ and $(L',\leq,B')$ be effective lattices, then a map $f : L\to L'$ is *computable* provided that it is continuous and the relation $\{(n,m)\in N\times N : b'_m \ll f(b_n)\}$ is recursively enumerable.

As an example, a function $f : U \to U$ is computable if and only if f preserves the last upper bounds and the relation $\{(p,q)\in U_Q\times U_Q : p<f(q)\}$ is recursively enumerable.

Proposition 3.7. Let $(L,\leq,B)$ and $(L',\leq,B')$ be based lattices and $f : L\to L'$ a map, then the following are equivalent:

i) f is continuous.

ii) $f(x) = Sup\{f(b) : b\in B, b\ll x\}$.

Consequently, if $f : L\to L'$ is a computable function, then

$$x \text{ semi-decidable} \Rightarrow f(x) \text{ semi-decidable}.$$

4. Decidable elements

To define the notion of decidability we need to dualize some of the notions in the previous sections. Given an ordered set $(D,\leq)$, we denote by $(D,\leq_d)$ its dual, i.e. the ordered structure obtained by setting $x \leq_d y$ provided that $x \geq y$. Any order-theoretical notion in $(D,\leq)$ is associated with its dual, i.e. the same notion interpreted in $(D,\leq_d)$. As an example, the dual of the notion of upward directed family, is the notion of is *downward directed* family. So $(x_i)_{i\in I}$ is downward directed if for any x_i, x_j there is x_t such that $x_t\leq x_i$ and $x_t\leq x_j$. We say that y is *way above* x and we write $x\ll^d y$ in the case y is way below x in $(L,\leq_d)$. Then $x\ll^d y$ provided that, for every downward directed subset A

$$x \geq Inf\,A \Rightarrow \text{there is } a\in A \text{ such that } y \geq a.$$

Obviously

$$x\ll^d y \Rightarrow x\leq y$$

and, if L is a finite chain,

$$x\ll^d y \Leftrightarrow x\leq y.$$

If L coincides with U, then

$$x\ll^d y \Leftrightarrow \text{either } y = 1 \text{ or } x<y.$$

Definition 4.1. A structure $(L, \leq,B, \underline{B})$ is called a *reversible* (*effective*) *continuous lattice* provided that both the structures $(L,\leq,B)$ and $(L,\leq_d,\underline{B})$ are based (effective) continuous lattices. In such a case we say that $B = (b_n)_{n\in N}$ is the *basis* and $\underline{B}= (\underline{b}_n)_{n\in N}$ the *dual basis* of $(L, \leq,B, \underline{B})$.

Obviously,

$$x = Sup\{b\in B : b\ll x\} = Inf\{\underline{b}\in\underline{B} : x\ll^d \underline{b}\},$$

for any $x\in L$. So, in a sense, in a reversible continuous lattice it is possible to approximate any element both from below and from above.

Definition 4.2. Given a reversible effective lattice $(L, \leq,B, \underline{B})$, we say that x is *decidable* provided that x is semi-decidable both in $(L, \leq, B)$ and $(L, \leq_d, \underline{B})$, i.e. if both the cuts

$$\{n\in N : b_n\ll x\} \quad ; \quad \{n\in N : x\ll^d b_n\}$$

are recursively enumerable.

Trivially, 0 and 1 are decidable elements in any reversible effective lattice. The proof of the following proposition is immediate.

Proposition 4.3. Given an element x of a reversible effective lattice, the following are equivalent:

i) x is decidable

ii) two total recursive functions $h : N \to N$ and $k : N \to N$ exist such that $(b_{h(n)})_{n\in N}$ is order preserving, $(\underline{b}_{k(n)})_{n\in N}$ is order reversing and

$$Sup_{n\in N} b_{h(n)} = x = Inf_{n\in N}\underline{b}_{k(n)}.$$

iii) a nested, effectively computable sequence $([b_{h(n)}, \underline{b}_{k(n)}])_{n\in N}$ of intervals exists such that

$$\{x\} = \cap_{n\in N} [b_{h(n)}, \underline{b}_{k(n)}]$$

An easy way to obtain reversible continuous lattices is by the notion of *involution*, i.e. a map $- : L \to L$ such that

i) $-0 = 1$; $-1 = 0$.
ii) $-(x\vee y) = -x\wedge -y$; $-(x\wedge y) = -x\vee -y$.
iii) $-(-(x)) = x$.
Observe that an involution is an isomorphism $-: L \to L_d$ among L and its dual L_d.

Definition 4.4. A structure $(L, \leq, -, B)$ is an *effective lattice with an involution* if $(L, \leq, B)$ is an effective lattice and - is an involution such that $\{(n,m)\in N\times N : -b_n \ll -b_m\}$ is recursively enumerable.

Let $L = \{x_1,\ldots,x_n\}$ be a finite lattice where $0 = x_0<\ldots<x_n = 1$. Then there is a unique involution defined by setting $-x_i = x_{n-i}$. In the case of the interval U, an involution is obtained by setting $-x = 1-x$.

Since an involution is an isomorphism and in accordance with the fact that any isomorphism preserves the definable relations, we have the following:

Proposition 4.5. Let L be a lattice with an involution. Then, for any $x\in L$,

$$x \ll^d y \Leftrightarrow -y \ll -x.$$

Any effective lattice with an involution defines a reversible effective lattice.

Proposition 4.6. Let $(L, \leq, B, -)$ be an effective lattice with an involution and set $\underline{B} = (\underline{b}_n)_{n\in N}$ where $\underline{b}_n = -b_n$. Then $(L,\leq,B, \underline{B})$ is a reversible effective lattice such that all the elements in B and in $\underline{B}$ are decidable and

x is decidable $\Leftrightarrow$ both x and $-x$ are semi-decidable.

This proposition entails that any finite chain L is a reversible effective lattice in which $B = \underline{B} = L$. In this lattice all the elements are decidable. The interval U is a reversible effective lattice in which $B = \underline{B} = U_Q$. In this lattice are decidable all the elements x such that both the sections $\{r \in U_Q : r < x\}$ and $\{r \in U_Q : x<r\}$ are recursively enumerable, i.e. the decidable elements coincide with the recursive real numbers.

5. Direct products

As observed in the introduction, the pieces of information we have to consider in a fuzzy logic are L-subsets of formulas, i.e. elements of a direct power of L. This leads to define the notion of direct products of effective continuous lattices.

Proposition 5.1. Let $(L_i,\leq_i,B_i)_{i\in S}$ be a family of based lattices. Then we obtain a based lattice $(\Pi_{i\in S}L_i,\leq,B)$ where $(\Pi_{i\in S}L_i,\leq)$ is the direct product of the family $((L_i,\leq_i))_{i\in I}$ of lattices and

$$B = \{f \in \Pi_{i\in S}B_i : Supp(f) \text{ is finite}\}. \tag{5.1}$$

In such a lattice, for any f and g in $\Pi_{i\in S}L_i$,

$$f \ll g \Leftrightarrow Supp(f) \text{ is finite and } f(j) \ll g(j) \text{ for any } j\in Supp(f). \tag{5.2}$$

Definition 5.2. Let $(L_i,\leq_i,B_i)_{i\in S}$ be a family of based lattices. Then we call *direct product* of such a family the based lattice defined in Proposition 5.1. If all the elements in the family coincide with the lattice $(L,\leq,B)$, then we call *direct power of* $(L,\leq,B)$ *with index set S* such a direct product.

Note that the direct product of $(L_i,\leq_i,B_i)_{i\in S}$ as defined in universal algebra is the structure $(\Pi_{i\in S}L_i,\leq, \Pi_{i\in S}B_i)$ and therefore, in the case S infinite, is different from the just given notion. By dualizing the proof of Proposition 5.1, we obtain the proof of the following proposition.

Proposition 5.3. Let $((L_i,\leq_i,B_i, \underline{B}_i))_{i\in S}$ be a family of reversible lattices. Then we obtain a reversible lattice $(\Pi_{i\in S}L_i,\leq,B,\underline{B})$ by assuming that $(\Pi_{i\in S}L_i,\leq,B)$ is the direct product of $((L_i,\leq_i,B_i))_{i\in S}$ and

$$\underline{B} = \{f \in \Pi_{i\in S}\underline{B}_i : Cosp(f) \text{ is finite}\}. \tag{5.4}$$

In such a lattice,

$$f \ll^d g \Leftrightarrow Cosp(f) \text{ is finite and } f(j) \ll^d g(j) \text{ for any } j\in S. \tag{5.5}$$

The definition of the direct product of a family of effective lattices is a little more complicate and we have to confine ourselves only to the *uniformly effective* families $(L_i,\leq_i,B_i)_{i\in S}$ of effective continuous lattices. This means that we have to assume that $\{(n,m,i)\in N^3 : b_n \ll_i b_m\}$ is recursively enumerable and that two recursive maps *join* : $N\times N\times N\to N$ and *meet* : $N\times N\times N\to N$ exist such that

$$b(i,n)\vee b(i,m) = b_{join(i,n,m)} \ , \ b(i,n)\wedge b(i,m) = b_{meet(i,n,m)}$$

where $b(i,n)$ is the n-element of B_i.

Proposition 5.4. Let S be a set admitting a code, then the direct product of an uniformly effective family of continuous lattices $(L_i,\leq_i,B_i)_{i\in S}$ is an effective continuous lattice.

In accordance with such a proposition, the product of a finite number of effective lattices $(L_1,\leq_1,B_1)$, $(L_2,\leq_2,B_2),\ldots,(L_k,\leq_k,B_k)$ is an effective lattice with basis $B_1\times\ldots\times B_k$. Moreover,

$$(\lambda_1,\ldots,\lambda_k)\ll(\mu_1,\ldots,\mu_k) \Leftrightarrow \lambda_1\ll\mu_1,\ldots,\lambda_k\ll\mu_k.$$

Also, let $(L,\leq B)$ be an effective lattice and denote by $b(i,j)$ the j-element in B_i. Then a several variable map $f : L_1\times\ldots\times L_k \to L$ is *computable* provided that it is continuous with respect each variable and $\{(n_1,n_2,\ldots,n_k,m)\in N^k\times N : b_m \ll f(b(1,n_1),\ldots,b(k,n_k)\}$ is recursively enumerable.

Proposition 5.5. The composition of computable maps is a computable map. Namely, let $h : L^t\to L$ and $g_1 : L^k \to L,\ldots, g_t : L^k \to L$ be computable maps and let $f : L^k \to L$ be the map such that $f(x_1,\ldots,x_k) = h(g_1(x_1,\ldots,x_k),\ldots,g_t(x_1,\ldots,x_k))$ for any $x_1,\ldots,x_k$ in L. Then f is computable.

Analogous definitions and results can be given in the case of the reversible effective lattices and in the case of effective lattices with an involution.

6. Effective lattice of the *L*-subsets of a given set

Given a set S, we call *L-subset* of S any element in the direct power L^S, i.e. any map $s : S \to L$ from S into L. We interpret an L-subset as a generalized characteristic function to represent the extension of a vague predicate. The usual interpretation is that L is the set of truth values of a multi-valued logic and, for every $x\in S$, $s(x)$ is the *membership degree* of x to s. In this paper we are interested also to interpret the elements in L as pieces of information about the truth values and therefore to interpret $s(x)$ as a constraint on the degree of membership of x to s. So, the usual notion of subset is extended into two directions, we admit different levels of membership degree and we admit incomplete pieces of information about these membership degree. We denote by $\cup$ and $\cap$ the lattice operations in L^S, i.e. the operations defined by setting, for any $s_1, s_2 \in L^S$ and $x\in S$

$$(s_1\cup s_2)(x) = s_1(x)\vee s_2(x) \ ; (s_1\cap s_2)(x) = s_1(x)\wedge s_2(x).$$

In an analogous way we define the infinitary unions and intersections. Given $X\in P(S)$, the *characteristic function* of X is the map $c_X : S \to L$ defined by setting $c_X(x) = 1$ if $x\in X$ and $c_X(x) = 0$ otherwise. We call *crisp* an L-subset s such that $s(x)\in\{0,1\}$ for every $x\in S$ and we can identify the classical subsets of S with the crisp L- subsets of S via the characteristic functions. Indeed, the map $H : P(S)\to L^S$ defined by setting $H(X) = c_X$ for every $X\in P(S)$ is a complete embedding of the lattice $(P(S), \cup, \cap,\emptyset,S)$ into the lattice $(L^S, \cup, \cap, c_\emptyset, c_S)$. Given $x \in S$ and $\lambda\neq 0$, the *λ-singleton* is the L-subset $\{x\}^\lambda$ defined by setting $\{x\}^\lambda(z) = \lambda$ if $z = x$ and $\{x\}^\lambda = 0$ otherwise. We say that s is *finite* provided that $Supp(s)$ is finite, i.e. s is a finite union of singletons. We say that s is *co-finite* provided that $Cosp(s)$ is finite. We call finite also the empty set $\varnothing$ and co-finite the whole set S. The classes of finite and co-finite L-subsets of S are denoted by $Fin(L^S)$ and $Cof(L^S)$, respectively. Assume that in L an involution - : $L\rightarrow L$ is defined, then we call *complement* of s the L-subset $-s$ defined by setting $(-s)(x) = -s(x)$. Obviously, in such a case an L-subset s is finite if and only if $-s$ is co-finite.

As a particular case of the propositions in Section 5, we obtain the following ones.

Proposition 6.1. Let $(L,\leq,B)$ be an effective lattice and S be a nonempty set admitting a code. Then the class L^S of L-subsets of S is an effective lattice admitting as a basis the class $Fin(B^S)$ of finite L-subsets of S with values in B. Also, for any s_1 and s_2 in L^S, we have that

$$s_1 \ll s_2 \;\Leftrightarrow\; s_1 \text{ is finite and } s_1(x)\ll s_2(x) \text{ for every } x\in S.$$

Observe that, by definition, an L-subset s is semi-decidable provided that the set

$$\{n\in N : b_n\ll s\} = \{n\in N : b_n(i)\ll s(i) \text{ for any } i\in Supp(b_n)\}$$

is recursively enumerable.

Proposition 6.2. Let $(L,\leq,B,\underline{B})$ be a reversible lattice and S be a nonempty set admitting a code. Then L^S is reversible with dual basis the class $Cof(B^S)$ of co-finite L-subsets of S with values in B. If $(L,\leq,B,-)$ is with an involution, then L^S is an effective lattice with the complement as an involution.

If L is reversible , s is decidable if and only if both the sets

$$\{n\in N : b_n(i)\ll s(i) \text{ for any } i\in Supp(b_n)\} \;\; ; \; \{n\in N : s(i)\ll^d \underline{b}_n(i) \text{ for any } i\in Cosp(\underline{b}_n)\}$$

are recursively enumerable. The following is a simple characterization of the semi-decidable and the decidable L-subsets.

Proposition 6.3. Let $(L,\leq,B)$ be a effective continuous lattice and $s\in L^S$. Then s is semi-decidable if and only if its hypograph

$$H(s) = \{(x,\lambda) \in S\times B\text{: } \lambda\ll s(x)\} \tag{6.1}$$

is recursively enumerable. Let $(L,\leq,B,\underline{B})$ be a reversible effective lattice. Then s is decidable if and only if both $H(s)$ and the dual hypograph

$$K(s) = \{(x,\lambda) \in S\times \underline{B}\text{: } s(x)\ll^d \lambda\} \tag{6.2}$$

are recursively enumerable.

Proposition 6.4. Let s be a crisp L-subset. Then s is semi-decidable if and only if s is (the characteristic function of) a recursively enumerable subset of S.

Proposition 6.5. A continuous map $D : L^S\rightarrow L^S$ is computable if and only if the relation $\lambda\ll D(b)(x)$ is recursively enumerable.

By extending the classical notion of m-reducibility, we say that s_1 is *m-reducible* to s_2, and we write $s_1\leq_m s_2$, provided that a recursive map $h : S\rightarrow S$ exists such that, for any $x\in S$

$$s_1(x) = s_2(h(x)).$$

We call *universal* any maximum with respect to $\leq_m$ in the class of semi-decidable L-subsets.

Proposition 6.6. The relation $\leq_m$ is a pre-order. Assume that $s_1\leq_m s_2$, then

$$s_2 \text{ semi-decidable} \Rightarrow s_1 \text{ semi-decidable},$$

$$s_2 \text{ decidable} \Rightarrow s_1 \text{ decidable}.$$

Moreover, an universal L-subset exists.

7. The main cases

In this section we consider some examples which are the basic ones in fuzzy logic.

Proposition 7.1. Let L be a finite chain. Then the class L^S of L-subsets of S is an effective lattice with an involution and therefore a reversible effective lattice. Its basis is the class $Fin(L^S)$ of finite L-subsets of S, its dual basis is the class $Cof(L^S)$ of co-finite L-subsets of S. Also

$$s_1 \ll s_2 \;\Leftrightarrow\; s_1 \subseteq s_2 \text{ and } s_1 \text{ is finite}$$

and

$$s_1 \ll^d s_2 \Leftrightarrow s_1 \subseteq s_2 \text{ and } s_2 \text{ is co-finite}$$

In particular, the class $P(S)$ of subsets of S is an effective lattice with an involution whose basis is the class of finite subsets of S and whose dual basis is the class of co-finite subsets. Also

$$X \ll Y \Leftrightarrow X \subseteq Y \text{ and } X \text{ is finite}$$

and

$$X \ll^d Y \Leftrightarrow X \subseteq Y \text{ and } Y \text{ is co-finite.}$$

Proposition 7.2. Let L be a finite chain. Then an L-subset s is semi-decidable if and only if there is a recursive function $h : S \times N \rightarrow L$ increasing with respect to the second variable such that

$$s(x) = Max_{n \in N} h(x,n).$$

An L-subset s is decidable if and only if s is a computable function from S to L.

Proposition 7.3. Let L be a finite chain. Then, the following are equivalent:
i) s is a recursively enumerable L-subset
ii) all the closed cuts $C(s,\lambda) = \{x \in S : s(x) \geq \lambda\}$ are recursively enumerable
iii) all the open cuts $O(s,\lambda) = \{x \in S : s(x) > \lambda\}$ are recursively enumerable.

The following corollary shows that, in the case of a finite chain L, the proposed notion of semi-decidable is the only possible extension of the classical one such that
- the constant L-subsets are semi-decidable
- the union of two semi-decidable L-subsets is semi-decidable
- the intersection of two semi-decidable L-subsets is semi-decidable.

Corollary 7.4. Let L be a finite chain, then the lattice of the semi-decidable L-subsets is the lattice generated by the recursively enumerable subsets and the constant L-subsets.

Now we will examine the case $L = U$. In such a case we call *fuzzy subsets* of S the U-subsets.

Proposition 7.5. The class of fuzzy subsets of S is an effective lattice with the complement as an involution. The basis is the class $Fin(U_Q{}^S)$ of finite fuzzy subsets of S with rational values. The dual basis is the class $Cof(U_Q{}^S)$ of co-finite fuzzy subsets of S with rational values. Moreover

$$s_1 \ll s_2 \Leftrightarrow s_1(x) < s_2(x) \text{ for every } x \in Supp(s_1) \text{ and } s_1 \text{ is finite.}$$

$$s_1 \ll^d s_2 \Leftrightarrow s_1(x) < s_2(x) \text{ for every } x \in Cosp(s_2) \text{ and } s_2 \text{ is co-finite.}$$

As the following proposition emphasizes, the proposed notions of semi-decidability and decidability for fuzzy subsets are in accordance with the definitions given in Biacino and Gerla 1989.

Proposition 7.6. A fuzzy subset s is semi-decidable if and only if there is a recursive function $h : S \times N \rightarrow U_Q$ increasing with respect to the second variable such that

$$s(x) = Sup_{n \in N} h(x,n).$$

A fuzzy subset s is decidable if and only if there is a nested computable sequence $([h(x,n), k(x,n)])_{n \in N}$ of intervals such that

$$\{s(x)\} = \cap_{n \in N}[h(x,n), k(x,n)].$$

8. Effective inferential apparatus in fuzzy logic

Let F be a nonempty set whose elements we call *formulas* and V be a complete lattice whose elements we interpret as truth values. Then an *abstract V-semantics*, in brief a *semantics*, is a class M of V-subsets of formulas. We call a *model* or an *interpretation* any element m in M and we interpret $m(\alpha)$ as the truth value of α in m. As an example, we can assume that M is the set of truth-functional valuations of a multi-valued logic. We call *constraint frame* any closure system in V, i.e. any class L of subsets of V which is closed with respect to the intersections and containing V. Given $X \subseteq V$, we denote by $<X>$ the element in L *"generated"* by X, i.e. the intersection of all the elements in L containing X. We consider L as a complete lattice with the reverse of the inclusion relation. This since the order we are interested in is with respect to the information content. In L the join of a family $(X_i)_{i\in I}$ is the intersection $\cap_{i\in I}X_i$ and the meet is $<\cup_{i\in I}X_i>$. Given an L-subset s of formulas, we say that $m\in M$ is a *model of s*, in brief $m \vDash s$, provided that $m(\alpha)\in s(\alpha)$ for any $\alpha\in F$. In other words, m is a model of s provided that for any formula α the truth value $m(\alpha)$ of α in m satisfies the constraint $s(\alpha)$. Two L-subsets of formulas are *logically equivalent* provided that they have the same models. We can associate any model m with an L-subset of formulas τ_m obtained by setting $\tau_m(\alpha) = <\{m(\alpha)\}>$. Trivially, $m \vDash s$ if and only if $\tau_m\geq s$. Any L-semantics defines a *logical consequence operator* $L_c : L^F\to L^F$ obtained by setting

$$L_c(s)(\alpha) = \cap\{\tau_m(\alpha) : m \vDash s\} = \cap\{\tau_m(\alpha) : \tau_m \geq s\}.$$

The proof of the following proposition is immediate.

Proposition 8.1. L_c is a closure operator, i.e.

$$i)\ L_c(s) \supseteq s \quad ; \quad ii)\ s_1 \subseteq s_2 \Rightarrow L_c(s_1) \subseteq L_c(s_1) \quad ; \quad iii)\ L_c(L_c(s)) = L_c(s).$$

The intersection of a family of theories is a theory. Moreover, two L-subsets of formulas s_1 and s_2 are logically equivalent if and only if $L_c(s_1)=L_c(s_2)$.

We interpret an L-subset s of formulas as a L-subset of hypotheses (the available information), and $L_c(s)$ as the L-subset of consequences from s. We call *theory* any fixed point of L_c. Then, for any s, $L_c(s)$ is a theory, we call *the theory generated by s.* For any model m, τ_m is an example of a theory.

Definition 8.2. Let V be a complete lattice, and L be a constraint frame. Then a *abstract deduction L-system* is a pair (L^F,D) where D is a continuous closure operator.

A more adequate definition of deduction system taking in account of the effectiveness of the inferential processes is the following one.

Definition 8.3. Let V be a complete lattice and L be a constraint frame in V such that $(L,\leq,B)$ is an effective lattice with respect to a suitable basis B. Then an *effective abstract deduction L-system* is an abstract deduction system (L^F,D) such that D is computable in the effective lattice $(L^F,\leq,Fin(B^F))$. An *effective L-logic* is a structure (L^F,D,M) such that M is an abstract L-semantics and (L^F,D) is an effective abstract deduction L-system such that $D = L_c$.

In this paper we are not interested in the semantics part of the L-logics and therefore we concentrate our attention on the effective deduction systems. If we interpret the relation $b\ll s$ as *b is a manageable piece of information of s*, then, the continuity and the computability hypotheses means that the theory generated by s is obtained in an effective way from manageable pieces of information in s. This means that, if we call *axiomatizable* a theory τ generated by a semi-decidable L-subset of hypotheses, then the following holds true.

Theorem 8.4. *Let (L^F,D) be a deduction L-system. Then any axiomatizable theory is semi-decidable.*

Such a theorem looks be a reason in favour of a domain-based theory of computability for fuzzy logic. Note that, in accordance with Gerla 2001, given a multi-valued logic we can define the L-subset $taut : F \to L$ of tautologies (the a-priori constraints on the truth values of the formulas, in a sense). Then, in the case such a logic is axiomatizable, *tau* is semi-decidable. Nevertheless, this does not entails that the set $\{\alpha\in F : tau(\alpha) = 1\}$ of "*tautologies"* is semi-decidable. Indeed, as a matter of fact, the following proposition holds true which is an immediate consequence of basic and well known results in literature (see Scarpellini 1962, Hájek 1998, Montagna 2001).

Theorem 8.5 *A subset of S is a closed cut of a recursively enumerable fuzzy subset iff it belongs to the Π_2-level of the arithmetical hierarchy.*

The following proposition emphasizes the logical interpretation of the deduction L-systems.

Proposition 8.6. An abstract deduction L-system (L^F,D) is effective if and only if there is a recursive (recursively enumerable) relation $b \vdash_{\pi}^{\lambda} \alpha$ where $b\in Fin(B^F)$, $\pi\in N$, $\alpha\in F$ and $\lambda\in B$, such that

$$b \vdash_{\pi}^{\lambda} \alpha\,,\; b \vdash_{\pi'}^{\lambda'} \alpha \;\Rightarrow\; \text{there is } \pi'' \text{ such that } b \vdash_{\pi''}^{\lambda\vee\mu} \alpha \tag{8.2}$$

and

$$D(s)(\alpha) = Sup\{\lambda\in B : \text{there is } b\ll s \text{ and there is } \pi\in N \text{ such that } b \vdash_{\pi}^{\lambda} \alpha\}. \tag{8.3}$$

We interpret an element $\pi \in N$ as the code number of a proof in a deduction apparatus and the recursive relation $b \vdash_{\pi}^{\lambda} \alpha$ as "π is a proof from b that α satisfies the constraint λ ". This justifies condition (8.2). The recursiveness of the relation $b' \vdash_{z}^{\lambda} \alpha$ expresses the fact that we are able to decide if something is a proof of a formula α from a finite piece of hypotheses and also to calculate the information λ given by such a proof.

9. Hilbert deduction systems.

The following is a less abstract definition of deduction apparatus for a fuzzy logic (see for example Gerla 2000).

Definition 9.1. Let L be a complete lattice. Then an *Hilbert L-system* is a pair $S = (a, INF)$ where $a : F\to L$ is an L-subset of F, the *L-subset of logical axioms*, and INF is a set of L-inference rules. In turn, an *L-inference rule* is a pair $r = (r',r'')$, where

- r' is a partial n-ary operation on F whose domain we denote by $Dom(r)$,
- r'' is an n-ary operation on L which is continuous in each variable, i.e.

$$r''(x_1,..., x, ..., x_n) = Sup\{r''(x_1, ...,x', ..., x_n) : x' \ll x\}. \tag{9.1}$$

In other words, an inference rule r consists

- of a *syntactical component* r' that operates on formulas (i.e. an inference rule in the usual sense),
- of a *valuation component* r'' that operates on constraints on the truth-values to calculate how we can obtain information about the conclusion from the available information on the premises (see [12], [27] and [18]).

A *proof* π of α is a sequence $\alpha_1,...,\alpha_m$ of formulas such that $\alpha_m = \alpha$ together with a sequence of related *"justifications"*. This means that, given any formula α_i, we must specify whether

(i) α_i is assumed as a logical axiom; or
(ii) α_i is assumed as an hypothesis; or
(iii) α_i is obtained by a rule (we have to indicate also the rule and the formulas used to obtain α_i).

Differently from the crisp case, the justifications are necessary since different justifications of the same formula give rise to different pieces of information. Let s an L-subset of formulas. Then the constraint $Val(\pi,s)$ furnished by π on the truth value of α (given s) is defined by induction by setting

$$Val(\pi,s) = \begin{cases} a(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a logical axiom,} \\ s(\alpha_m) & \text{if } \alpha_m \text{ is assumed as an hypothesis,} \\ r''(Val(\pi_{s(1)},s),\dots,Val(\pi_{s(n)},s)) & \text{if } \alpha_m = r'(\alpha_{s(1)},\dots, \alpha_{s(n)}) \end{cases}$$

where π_i denotes the proof $\alpha_1,...,\alpha_i$. Now, it should be possible to find another proof π' of α such that $Val(\pi',v) > Val(\pi,v)$. This happens, for instance, if the assumptions used in π' are more informative than the assumptions used in π. In other words, unlike the usual Hilbert systems, different proofs of a same formula α can give different pieces of information on α. This suggests that, given an L-subset s of hypotheses (the available information), in order to evaluate α we must refer to the whole set of proofs of α.

Definition 9.2. Given an Hilbert L-system S, we call *deduction operator associated with S* the operator $D : L^F\to L^F$ defined by setting,

$$D(s)(\alpha) = Sup\{Val(\pi,s) : \pi \text{ is a proof of } \alpha\}, \tag{9.2}$$

for every L-subset s of formulas and every formula α.

The intended meaning is that $D(s)(\alpha)$ is the best "constraint" on the actual truth value of α we can draw from s.

Proposition 9.3. The deduction operator D of an Hilbert L-system is a continuous closure operator. Therefore any Hilbert L-system is associated with an abstract deduction L-system (L^F,D).

It is interesting to observe that τ is a theory of (L^F,D) if and only if τ contains the L-subset of logical axioms and it is closed with respect to any inference rule r, i.e.

$$\tau(r'(\alpha_1,...,\alpha_n)) \geq r''(\tau(\alpha_1),...,\tau(\alpha_n))$$

for any $(\alpha_1,...,\alpha_n)\in Dom(r)$. Since we are interested only on the operator D, it is not restrictive to assume that in an Hilbert system there is a *fusion rule*, i.e. a rule $r = (r',r'')$ such that

$$D_r = \{(\alpha,\alpha): \alpha\in F\} \ ; \ r'(\alpha,\alpha) = \alpha \ ; \ r''(\lambda_1,\lambda_2) = \lambda_1\vee\lambda_2.$$

This rule enables us to fuse two proofs π_1 and π_2 of a formula α into a new proof π obtained by concatenating π_1 with π_2 in such a way that $Val(\pi,s)=Val(\pi_1,s)\vee Val(\pi_2,s)$. This entails that the set $\{Val(\pi,b) : \pi$ is a proof of $\alpha\}$ is directed. Also, observe that if $s\supseteq a$, then we can confine ourselves only to the proofs in which there is no formula assumed as a logical axiom.

In this paper we will consider a notion of Hilbert L-system taking in account of the effectiveness of the inferential processes.

Definition 9.4. An Hilbert L-system $S = (a, INF)$ such that INF is finite is *effective* provided that:
(a) any inference rule $r = (r',r'')$ is computable, i.e. $Dom(r)$ is decidable, r' is recursive in D and r'' is computable in each variable
(b) the L-subset a of logical axioms is semi-decidable.

Proposition 9.5. The deduction operator D of an effective Hilbert L-system S is a computable closure operator. Therefore any effective Hilbert L-system is associated with an effective abstract deduction L-system (L^F,D).

In Biacino and Gerla 2002 one proves the converse of Proposition 9.3 and Proposition 9.5 in the case $L = U$. Therefore the abstract approach to fuzzy deduction is equivalent with the approach based on the Hilbert systems.

Proposition 9.6. Assume that $L = U$. Then any abstract (effective) deduction L-system is the deduction system of a suitable (effective) Hilbert L-system.

10. Deduction systems by interval-constraints

In Section 9 the lattice L is a class of constraints on the truth values of the formulas. A simple example is the following one. Let V be an effective lattice whose elements we interpret as truth values of a multi-valued logic. Then we set L equal to the class of closed interval in V, i.e.,

$$I(V) = \{[a,b] : a,b\in V, a\leq b\}\cup\{\varnothing\}.$$

In such a way we are able to consider a signed formula as $(\alpha,[0.3,0.5])$ to represent the information "the truth value of α lies between 0.3 and 0.5". Different choices are possible, obviously. As an example, should be interesting also to consider constraints, as

- the probability that the truth value of α is 0.7 is 0.8
- it is possible at degree 0.8 that the truth value of α is 0.7.

Obviously, in $(I(V),\leq)$ we have that,

$$[a,b]\leq[c,d] \Leftrightarrow [a,b]\supseteq[c,d] \Leftrightarrow a\leq c \text{ and } b\leq d \Leftrightarrow a\leq c \text{ and } b\leq_d d.$$

$$([x_i,y_i])_{i\in I} \text{ upward directed } \Leftrightarrow (x_i,)_{i\in I} \text{ upward directed and } (y_i)_{i\in I} \text{ downward directed.}$$

$$([x_i,y_i])_{i\in I} \text{ downward directed } \Leftrightarrow (x_i,)_{i\in I} \text{ downward directed and } (y_i)_{i\in I} \text{ upward directed.}$$

The maximum is $\varnothing$ and the minimum is $[0,1]$ while the maximal elements are the intervals $[x,x] = \{x\}$. In particular, if V is a finite chain,

$$[a,b]\ll[c,d] \Leftrightarrow a\leq c \text{ and } b\geq d \Leftrightarrow [c,d]\subseteq [a,b] \Leftrightarrow [a,b]\leq[c,d],$$

If $V = U$, we have to distinguish several cases. Indeed, we have that $[0,1]\ll[0,1]$,

$$[0,b]\ll[0,d] \Leftrightarrow d<b \Leftrightarrow [0,b)\supseteq[0,d]$$

$$[a,1]\ll[c,1] \Leftrightarrow a<c \Leftrightarrow (a,1]\supseteq[c,1]$$

and, in the remaining cases,

$$[a,b] \ll [c,d] \Leftrightarrow a<c \text{ and } b>d \Leftrightarrow (a,b) \supseteq [c,d].$$

Proposition 10.1. Assume that $(V,\leq,B,\underline{B})$ is an effective reversible lattice and set

$$I(B,\underline{B}) = \{[x,y] : x\in B, y\in\underline{B}\}.$$

Then $(I(V),\leq, I(B,\underline{B}))$ is an effective lattice such that, for any pair of nonempty intervals $[a,b]$ and $[c,d]$,

$$[a,b] \ll [c,d] \Leftrightarrow a\ll c \text{ and } d\ll^{d}b. \tag{10.1}$$

Proposition 10.2. Let $(V,\leq,B,\underline{B})$ be an effective reversible lattice and $(I(V),\leq,I(B,\underline{B}))$ be the associated effective lattice. Then the following are equivalent:

i) the interval $[a,b]$ is semi-decidable

ii) a is semi-decidable and b co-semi-decidable.

iii) a nested, effectively computable sequence $([b_{h(n)}, \underline{b}_{k(n)}])_{n\in N}$ of intervals exists such that

$$[a,b] = \cap_{n\in N} [b_{h(n)}, \underline{b}_{k(n)}]$$

iv) an effectively computable sequence $([b_{h(n)}, \underline{b}_{k(n)}])_{n\in N}$ of intervals exists such that

$$[a,b] = \cap_{n\in N} [b_{h(n)}, \underline{b}_{k(n)}]$$

v) two total recursive functions $h : N \to N$ and $k : N \to N$ exist such that $(q_{h(n)})_{n\in N}$ is order preserving, $(q_{k(n)})_{n\in N}$ is order reversing and

$$a = Sup_{n\in N} b_{h(n)} \ ; \ b = Inf_{n\in \underline{N}} \underline{b}_{k(n)}.$$

vi) two total recursive functions $h : N \to N$ and $k : N \to N$ exist such that

$$a = Sup_{n\in N} b_{h(n)} \ ; \ b = Inf_{n\in \underline{N}} \underline{b}_{k(n)}.$$

In particular, we have that a degenerate interval $\{x\}$ is semi-decidable in $(I(V),\leq,I(B,\underline{B}))$ if and only if it is decidable in $(L,\leq,B,\underline{B})$. If $V=U$, then $[a,b]$ is semi-decidable if and only if a it limit of a increasing computable sequence of rational number and b is limit of a decreasing computable sequence of rational numbers.

Now, we are able to propose the main definitions in this paper where we identify any element λ of V with the one-element interval $[\lambda]$.

Definition 10.3. Let V be an effective reversible lattice whose elements are interpreted as truth values for a multi-valued logic. Then we call *effective abstract interval-based V-logic* any effective abstract logic $(I(V)^F,D,M)$ where $M\subseteq V^F$.

Observe that in a very large class of L-logics we can consider the complete lattice $I^+(V) = \{[\lambda,1] : \lambda\in V\}$ instead of $I(V)$. A basic advantage of $I^+(V)$ is that it is isomorphic with V via the map $f : V\to I^+(V)$ defined by setting $f(\lambda) = [\lambda,1]$. In accordance, in the sequel we denote by λ the interval $[\lambda,1]$ and we refer to V instead of $I^+(V)$. We emphasize that, in spite of the isomorphism, the intended interpretations of these lattices are different.

Definition 10.4. Let $(V,\leq,B, -)$ be an effective lattice with an involution and assume that a suitable computable map $\neg : F\to F$ is defined. Then we say that a semantics M is *balanced* provided that

$$m(\neg\alpha) = -m(\alpha)$$

for any $m\in M$ and $\alpha\in F$.

The following proposition shows that in the case of balanced semantics the lattice $I^+(V)$ has the same expressive power than $I(V)$.

Proposition 10.5. Let M be a balanced semantics, then, for any $I(V)$-subset s of formulas there is a logically equivalent $I^+(V)$-subset of formulas s^+.

Definition 10.6. We call *effective abstract lower-constraints V-logic*, in brief *lower-constraints V-logic*, any effective abstract logic $(I^+(V)^F,D,M)$ whose semantics is balanced.

In such a logic we say that α is *decidable in a theory* τ provided that $\tau(\neg\alpha) = -\tau(\alpha)$.

Proposition 10.7. In a lower-constraints V-logic the following are equivalent
i) τ is a model
ii) τ is a complete theory
iii) any formula is decidable in τ.

The choice of considering $I^+(V)$ is the one usually adopted by fuzzy logic and it is shared by classical logic. In fact, in classical logic in representing the available information by a set T of formulas (the set of proper axioms), it is intended that the formulas in T are true and that no information we have about the remaining formulas. If we will claim that a formula α is false, then we put in T its negation $\neg\alpha$. So, in terms of constraints on the truth values of the formulas, we admit only signed formulas as $(\alpha,[1,1]) = (\alpha,\{1\})$, claiming that the truth value of α is 1, and $(\alpha,[0,1]) = (\alpha,\{0,1\})$, claiming that we have no information about the truth value of α. This means that we identify T with the map $s : F\rightarrow \{\{1\},\{0,1\}\}$ defined by setting $v(\alpha) = \{1\}$ if $\alpha\in T$ and $v(\alpha) = \{0,1\}$ otherwise.

11. Decidable theories, complete theories: some difficulties.
A basic property in classical logic is that any axiomatizable and complete theory is decidable. Now, there is a natural definition of completeness for an abstract logic. Indeed, we can call *complete* any theory τ which is maximal in the class of theories. This means that we cannot extend τ in a consistent way by adding new information. As an example, in the case of $I(V)$ this is attained by assuming that, for any formula α, $\tau(\alpha)$ is a one element interval, i.e., a truth value of α. Instead it is rather difficult to give the notion of decidable theory. For example, in $I(V)^F$ this is not possible since we cannot define in $I(V)^F$ a structure of reversible lattice, in general.

Proposition 11.1. The lattice $(I(V),\leq)$ is reversible if and only if V is a finite chain. It is possible to define in $(I(V),\leq)$ an involution if and only if $V = \{0,1\}$.

A way to face this difficulty is to confine ourselves to logics with a negation. In fact in such a case we refer to the lattice $I^+(V)$ and, due to the isomorphism with V, such a lattice is an effective lattice with an involution.

Theorem 11.2. Let τ be an axiomatizable and complete theory in a logic with a negation based on $I^+(V)$. Then τ is decidable.

This solution is not completely satisfactory. In fact, there are several interesting multi-valued logic which are not "with a negation" in the sense proposed in this paper. Moreover, also in the case of logic with a negation (in particular in classical logic) in considering $I^+(V)$ we confine ourselves to manage only "positive" information. Instead, in my opinion, should be interesting to examine the possibility of a symmetric approach to deduction in which both positive and negative information is managed.

A further tentative to give an answer to this question is suggested by bilattice theory. Indeed, consider the product of a lattice $(V,\leq)$ with its dual $(V,\leq_d)$, i.e. the complete lattice $B(V) = (V\times V, \leq_t)$ where

$$(x,x') \leq_t (y,y') \Leftrightarrow x\leq y \text{ and } x'\geq y'.$$

In this lattice the minimum and the maximum are (0,1) and (1,0), respectively. The intended interpretation is that we assign to a statement α the value $(x,y) \in B(V)$ provided that x is (a measure of) the degree of belief in α and y is (a measure of) the degree of disbelief in α. In particular, the pair (0,0) indicates we have no evidence both for and against α (no information), (1,0) that α is true, (0,1) that α is false, (1,1) that we are in the inconsistent situation of having full evidence both for and against α. The related lattice operations are defined by setting, for any $x, x', y, y', z, z' \in V$,

$$(x,x')\wedge(y,y') = (x\wedge x',y\vee y'), \quad (x,x')\vee(y,y') = (x\vee x',y\wedge y').$$

There is some formal connection between the lattices $(V\times V, \leq_t)$ and $(I(V),\leq)$. Indeed, consider the map $h : I(V)\rightarrow V\times V$ defined by setting $h(\emptyset) = (1,0)$ and $h([a,b]) = (a,b)$ if $a\leq b$. Then h is an embedding of $(I(V),\leq)$ into $(V\times V, \leq_t)$. Nevertheless, the meaning of the elements in $(I(V),\leq)$ is totally different from the meaning of the corresponding elements in $(V\times V, \leq_t)$. As an example, while in $(I(V),\leq)$ the interval

[0,1] represents the absence of information, in $B(V)$ the pair (0,1) represents the false truth value. The interest of $B(V)$ is that we can define an involution ~ in a natural way by switching the roles of belief and disbelief, i.e. by setting $\sim(x,x') = (x',x)$.

Proposition 11.3. Assume that $(V,\leq,B,\underline{B})$ is a reversible effective lattice and set $B^{\times} = B\times\underline{B}$. Then $(V\times V, \leq_t,B^{\times})$ is an effective lattice admitting ~ as an involution. Also
(x,y) semi-decidable in $(V\times V,\leq_t,I(B)) \Leftrightarrow x$ semi-decidable and y is co-semi-decidable in $(V,\leq,B)$,
(x,y) decidable in $(V\times V, \leq_t,I(B)) \Leftrightarrow x$ and y are both decidable in $(V,\leq,B)$.

Such a proposition show that we can define fuzzy logics based on $(V\times V, \leq_t,B^{\times})$ and, in accordance, to define a notion of decidable theory. Future works include an examination of such a possibility.

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